

Random Dirac Fermions and Non-Hermitian Quantum Mechanics

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We study the influence of a strong imaginary vector potential on the quantum mechanics of particles confined to a two-dimensional plane and propagating in a random impurity potential. We show that the wavefunctions of the non-Hermitian operator can be obtained as the solution to a two-dimensional Dirac equation in the presence of a random gauge field. Consequences for the localization properties and the critical nature of the states are discussed.

The field of non-Hermitian quantum mechanics has attracted great interest recently both in its connection to anomalous diffusion in random media [1], as well as to the statistical mechanics of flux lines in superconductors [2]. At the same time, motivated in part by the connection to properties of the integer quantum Hall transition and gapless superconductors detailed investigations have been made into the critical properties of Dirac Fermions coupled to random gauge fields [3–11]. The aim of this letter is to identify a connection between these problems which explains some of the unusual phenomena recently reported in the behavior of two-dimensional non-Hermitian random operators [2]. This correspondence is related to a *chiral symmetry* of effective Hamiltonians commonly used in the analysis of problems in non-Hermitian quantum mechanics.

The two-dimensional Hamiltonian we consider describes a particle propagating in a random scalar impurity potential, V and subject to a uniform imaginary vector potential, $i\mathbf{h}$

$$\hat{H} = \frac{1}{2m} (\hat{\mathbf{p}} + i\mathbf{h})^2 + V(\mathbf{r}). \quad (1)$$

The scalar potential, V is assumed to be real and drawn from some random distribution, $P[V]$ which, for now, is left unspecified. Related problems have been recorded in a variety of physical situations ranging from the study of reaction diffusion phenomena in biological systems [12], to advective diffusion in random media [1,13,14], and the study of fluctuating vortex lines in superconductors with columnar defects [2].

Remarkably, in contrast to properties of the Hermitian operator (i.e. one in which the vector potential is real), numerical studies [2] suggest localization properties of \hat{H} depend sensitively on the relative strength of \mathbf{h} . While, in dimensions $d \leq 2$, *all* wavefunctions of the Hermitian operator are believed to be localized [15], numerical evidence suggests that application of a sufficiently strong vector potential, \mathbf{h} induces a delocalization transition of states of the non-Hermitian operator. In contrast to the

situation in $1d$ [2,16], the mechanism and stability of the delocalization transition is not yet well understood.

Recent analytical studies have focussed on the spectral properties of the non-Hermitian Hamiltonian [17]. Treating the vector potential as a weak perturbation of the random Hamiltonian, impurity averaged properties of the Green function have been cast in the form of a functional field integral involving a supersymmetric non-linear σ -model. While capturing *universal* features of the complex spectrum of \hat{H} , evidence for delocalization of states was not sought. By contrast, in the present approach, we will impose a strong imaginary vector potential, and treat the random potential as a perturbation (a regime explored in $1d$ by Feinberg and Zee [16]). By doing so we will reveal an explicit connection between the Hamiltonian in Eq. (1) and the problem of Dirac fermions propagating in a random gauge field. In particular, we will find a regime of weak disorder in which it is possible to construct explicitly eigenfunctions of the non-Hermitian Hamiltonian \hat{H} for *individual* realizations of the disorder.

In the absence of the impurity potential, the eigenfunctions of \hat{H} are plane waves with complex eigenvalues, $z_0(\mathbf{p}) = (\mathbf{p}^2 - \mathbf{h}^2)/2m + i\mathbf{p} \cdot \mathbf{h}/m$. Since the Hamiltonian is real ($\hat{H} = \hat{H}^*$), eigenvalues occur in complex conjugate pairs, a property maintained in the presence of the random potential. In the infinite system, the spectrum forms a dense support occupying the region of the complex plane $|\text{Im } z| \leq \sqrt{2m\text{Re } z + |\mathbf{h}|^2} |\mathbf{h}|/m$. The two-dimensional density of states (DoS) takes the form

$$\nu_0(z) = \frac{\nu_0}{\pi} \frac{m}{[(2m\text{Re } z + \mathbf{h}^2)\mathbf{h}^2 - m^2(\text{Im } z)^2]^{1/2}}, \quad (2)$$

where ν_0 denotes the constant DoS of the Hermitian Hamiltonian.

Concerned with impurity averaged spectral properties of the Hamiltonian, we begin by defining the Green function $\hat{G}(z) = (z - \hat{H})^{-1}$ where z denotes the complex

argument. Using $\nu(z) \equiv (1/\pi\Omega)\text{tr } \partial_z^* G(z)$, where Ω represents the volume, $G(z)$ is shown to be non-analytic everywhere the DoS $\nu(z)$ is non-vanishing [14,18].

To properly account for non-analytic properties of the impurity averaged Green function, previous studies [14,17,18] have emphasized the need to express the Green function through a Hamiltonian which is explicitly Hermitian. This is achieved by constructing a matrix Hamiltonian with the 2×2 block structure

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & \hat{H} - z \\ \hat{H}^\dagger - z^* & 0 \end{pmatrix}. \quad (3)$$

In this representation, the Green function of the non-Hermitian operator is expressed as the off-diagonal element of the matrix Green function, $\hat{G}(z) = \lim_{\eta \rightarrow 0} \hat{G}_{21}(z)$, where, defining $\eta = 0^+$, $\hat{G} = (i\eta - \hat{\mathcal{H}})^{-1}$. Zero energy eigenstates of the matrix Hamiltonian $\hat{\mathcal{H}}$ yield eigenstates of the non-Hermitian Hamiltonian \hat{H} .

Defining Pauli matrices $\vec{\sigma}$ ($\sigma_0 = \mathbb{1}$) which operate in the 2×2 space, the *chiral* symmetry of the matrix Hamiltonian, $\hat{\mathcal{H}} = -\sigma_3 \hat{\mathcal{H}} \sigma_3$ implies that eigenvalues, ϵ_i of $\hat{\mathcal{H}}$ appear in pairs of opposite sign. Moreover, any such pair of eigenstates obeys $|\psi_{+\epsilon_i}\rangle = \sigma_3 |\psi_{-\epsilon_i}\rangle$.

The spectrum of $\hat{\mathcal{H}}$ depends sensitively on the strength of the vector potential, \mathbf{h} . In the absence of the random potential, the dispersion relation takes the form,

$$E_0(\mathbf{p}) = \pm |z_0(\mathbf{p}) - z|, \quad (4)$$

invariant under reflection about the axis parallel to \mathbf{h} . Thus, in contrast to a Hamiltonian involving a real vector potential, where the spectrum is described by a simple shift of the Fermi sphere, the continuous degeneracy of the zero eigenvalues (the poles of the Green function) is lifted. Instead, setting $\mathbf{h} \equiv h\mathbf{e}_2$, zero energy states exist at only *two* discrete points $\mathbf{p}_0^{(a)}$, $a = 1, 2$ [19] (see Fig. 1).

If the impurity potential is strong (i.e. $\ell \ll \hbar/h$, where ℓ denotes the transport mean free path associated with the random impurity potential, V), unperturbed states of the clean system are strongly mixed by the disorder (see Fig. 1). In this limit, the pole structure of the impurity averaged Green function, \hat{G} is smeared out. Correspondingly, statistical properties of $\hat{\mathcal{H}}$ are largely insensitive to the zero eigenvalues of $E_0(\mathbf{p})$. In this limit, one can expect the transport properties of the non-Hermitian operator to reflect those of the Hermitian counterpart. Conversely, if the impurity potential is weak ($\ell \gg \hbar/h$) the pole structure of the average Green function is dominated by the nature of the spectrum in the vicinity of the zeros, $\mathbf{p}_0^{(a)}$.

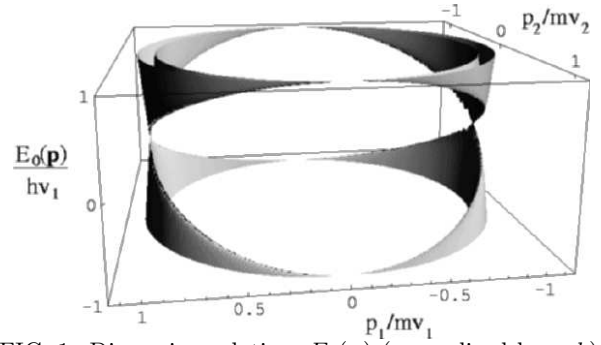


FIG. 1. Dispersion relation, $E_0(\mathbf{p})$ (normalized by $v_1 h$) in the vicinity of zero energy shown as a function of \mathbf{p} with $\mathbf{h} = \hbar \mathbf{e}_2$, $\text{Im } z = 0$ and $m v_1 / 2h = 10$. Note that the minimum scattering amplitude required to smear out the points of degeneracy is given by $\hbar/\tau \sim h v_1$.

Focusing on the limit of weak disorder

$$\text{Re } z \gg \frac{\hbar^2}{2m} \gg \frac{\hbar}{\tau}, \text{Im } z, \quad (5)$$

where τ represents the corresponding mean free scattering time of the random potential, a linearisation of the spectrum in the vicinity of the two zero eigenvalues separates the spectrum into two branches. Treating the random potential and $\text{Im } z$ as weak perturbations, and performing a gradient expansion in

$$\hat{\mathbf{p}}^{(a)} = \mathbf{p}_0^{(a)} + (-1)^a \hat{p}_1 \mathbf{e}_1 + \hat{p}_2 \mathbf{e}_2, \quad (6)$$

where $\mathbf{p}_0^{(a)} = (-1)^{a-1} m v_1 \mathbf{e}_1$, $m v_1 = (2m \text{Re } z + \hbar^2)^{1/2}$, and $m v_2 = h$, the low energy, long-wavelength expansion of the (unperturbed) Hamiltonian around the Fermi points $\mathbf{p}_0^{(a)}$ generates the *anisotropic* Dirac operator

$$\hat{\mathcal{H}}_D^{(0)} = -\tau_0 \otimes [\sigma_1 v_1 (\hat{\mathbf{p}} \cdot \mathbf{e}_1) + \sigma_2 v_2 (\hat{\mathbf{p}} \cdot \mathbf{e}_2)]. \quad (7)$$

Here we have introduced an additional set of Pauli matrices, $\vec{\tau}$ ($\tau_0 = \mathbb{1}$), that index the block structure associated with the reflection symmetry. The existence of two degenerate zero energy eigenstates of $\hat{\mathcal{H}}_D^{(0)}$ conspires with the exact chiral symmetry to yield an anisotropic Lorentz symmetry.

Being generally non-symmetric under reflection, matrix elements of the random impurity potential violate the reflection symmetry. Accounting for matrix elements which scatter across the Fermi surface (i.e. between the Dirac points) as well as within each subspace, the general Hamiltonian takes the form

$$\hat{\mathcal{H}}_D = \hat{\mathcal{H}}_D^{(0)} + \sum_{\nu=0}^3 \tau_\nu \otimes \sigma_1 V_\nu + \tau_0 \otimes \sigma_2 \text{Im } z, \quad (8)$$

where the impurity potentials, $V_\nu(\mathbf{r})$ are real random functions with Fourier components

$$\sum_\nu \tilde{V}_\nu(\mathbf{q}) \tau_\nu = \begin{pmatrix} \tilde{V}(-q_1, q_2) & \tilde{V}(q_1 - 2m v_1, q_2) \\ \tilde{V}^*(q_1 - 2m v_1, q_2) & \tilde{V}(+q_1, q_2) \end{pmatrix}.$$

Equations (7,8) represent an important intermediate result of this Letter: Firstly, the low energy sector of the original Hermitian Hamiltonian (3) has been described in terms of the stochastic Dirac Hamiltonian, $\hat{\mathcal{H}}_D$. Secondly, $\hat{\mathcal{H}}_D$ possesses the chiral symmetry $\sigma_3 \hat{\mathcal{H}}_D \sigma_3 = -\hat{\mathcal{H}}_D$, a direct consequence of the chirality of the auxiliary operator (3). The significance of the second observation lies in the fact that the behavior of chiral Dirac Hamiltonians can be analyzed *for individual realizations of the disorder*. As we shall see, it is possible to construct explicit solutions for the zero energy eigenfunction $|\Psi_D\rangle$ of $\hat{\mathcal{H}}_D$. From these the eigenfunctions $|\Psi_z\rangle$ ($|\Psi_{z^*}\rangle$) of the original non-Hermitian operator \hat{H} (\hat{H}^\dagger) can then be obtained as

$$|\Psi_z\rangle \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{\Pi}^\dagger |\Psi_D\rangle, \quad |\Psi_{z^*}\rangle \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{\Pi}^\dagger |\Psi_D\rangle, \quad (9)$$

for an *arbitrary* eigenvalue z provided Eq. (5) holds. Here the matrix structure refers to the σ -space and the operator $\hat{\Pi}$ accounts for the fact that in order to obtain eigenfunctions of the full problem, the eigenstates of the low energy expansion (8) have to be ‘boosted’ to the Fermi points $\mathbf{p}_0^{(a)}$, $a = 1, 2$, respectively [20].

We next turn to the explicit construction of the wavefunction $\Psi_D(\mathbf{r})$. To this end we first remove the anisotropy of the Hamiltonian (7) by rescaling the coordinates according to $r_\mu = (v_\mu/v)x_\mu$, $v = \sqrt{v_1 v_2}$. As a result $\hat{\mathcal{H}}_D$ takes the canonical form

$$\hat{\mathcal{H}}_D = \sum_{\mu=1}^2 \sigma_\mu \otimes (i\tau_0 \partial_\mu + \mathcal{A}_\mu),$$

where $\hbar = v = 1$ and the two components ($\mu = 1, 2$), $\mathcal{A}_\mu = A_\mu + B_\mu + C_\mu$ are given by

$$A_\mu = \delta_{\mu 1} \tau_0 V_0, \quad B_\mu = \delta_{\mu 1} \sum_{\nu=1}^3 \tau_\nu V_\nu, \quad C_\mu = \delta_{\mu 2} \tau_0 \text{Im } z.$$

The disorder in $\hat{\mathcal{H}}_D$ appears in the form of a minimal coupling to a generally non-abelian vector potential (i.e. $B_\mu \neq 0$). It is thus natural to seek a gauge transformation that removes the stochastic components of the Hamiltonian. Indeed, although the potentials, A_μ and B_μ are not in general of pure gauge type, the non-gauge components can be accounted for by extending the concept of gauge transformations so as to include ‘axial’ transformations.

Focusing on the abelian sector first, we decompose A_μ into a transverse (axial gauge) and a longitudinal (pure gauge) component: $A_\mu = \epsilon_{\mu\nu} \partial_\nu \chi_\perp + \partial_\mu \chi_\parallel$, respectively [21]. It is then straightforward to verify that

$$\Psi_D(\mathbf{x}) = e^{i\sigma_0 x_2 \text{Im } z} e^{i\sigma_0 \chi_\parallel(\mathbf{x})} e^{\sigma_3 \chi_\perp(\mathbf{x})} \begin{pmatrix} \Theta_+ \\ \Theta_- \end{pmatrix} \quad (10)$$

represents a solution of $\hat{\mathcal{H}}_D \Psi_D = 0$ for $B_\mu = 0$ [22]. Here, $\Theta_\pm \in \mathbb{C}$ and the gauge transformation $\exp(i\sigma_0 x_2 \text{Im } z)$ has been used to dispose of the small imaginary component of the eigenvalue z in the non-Hermitian problem. This contribution can be absorbed into the ‘boost’ component of $\hat{\Pi}$ (9) [23].

The treatment of the non-abelian components B_μ is conceptually similar but – due to their non-commutativity in τ -space – technically more involved. Referring to Ref. [24] for details, we merely state that in the presence of finite B_μ , Eq. (10) generalizes to

$$\Psi_D(\mathbf{x}) = e^{i\sigma_0 x_2 \text{Im } z} e^{i\sigma_0 \chi_\parallel(\mathbf{x})} U(\mathbf{x}) e^{\sigma_3 \chi_\perp(\mathbf{x})} e^{\sigma_3 \vec{\xi}(\mathbf{x}) \cdot \vec{\tau}} \times \begin{pmatrix} f(x_1 + ix_2) \Theta_+ \\ g(x_1 - ix_2) \Theta_- \end{pmatrix}. \quad (11)$$

Here, $U(\mathbf{x}) \in SU(2)$ plays the role of the abelian gauge transformation $\exp[i\sigma_0 \chi_\parallel(\mathbf{x})]$, whereas the vector field $\vec{\xi}$ plays the role of the abelian axial component χ_\perp and obeys an analogous but more complicated equation [24]. Finally, f and g represent analytic functions which are fixed by the boundary conditions (see below).

Equation (11) represents a non-perturbative solution to the zero energy Dirac equation for *any* given realization of the disorder. It, therefore, allows the construction of the eigenfunctions of the non-Hermitian Hamiltonian according to Eq. (9). The most important properties of Ψ_D are a) that the generalized gauge factors depend in a non-local way on the spatial distribution of the disorder potential and b) that the axial gauge factors lead to an exponential disorder dependent amplification of the *modulus* of the wavefunction, $\Psi_D^\dagger \Psi_D \equiv |\Psi_D|^2 = \Theta^\dagger e^{2\sigma_3 [\chi_\perp + \vec{\xi} \cdot \vec{\tau}]} \Theta$.

What can be said about the asymptotic behavior of the wavefunction Ψ_D , and, in particular, about its localization properties? To address this question we consider the impurity averaged two-point correlation function, $C(\mathbf{r}_1 - \mathbf{r}_2) \equiv |\Psi_D(\mathbf{r}_1)|^{2q_1} |\Psi_D(\mathbf{r}_2)|^{2q_2}$. From the structure of the solution we infer that $C = F_A \times F_B$ factorizes into an abelian and a non-abelian component both of which can be straightforwardly extracted from Eq. (11). To say more, it is necessary to specify both the form of the random potential distribution function, and the topology of the system. Here we focus on the thermodynamic limit with Gaussian white-noise distributed disorder, $P[V]$ of uniform variance. Under these circumstances, taking $f = g = 1$ in Eq. (11) represents the only admissible choice [25].

Defining $g_A = (2 \times 2\pi\nu\tau)^{-1}$ as the variance of the coarse-grained Gaussian distribution for A_μ (i.e. $\chi_\perp(\mathbf{x})\chi_\perp(\mathbf{y}) \propto -g_A \ln|\mathbf{x} - \mathbf{y}|$), and focusing, for simplicity, on the case $q_1 = q_2 = 1$, the impurity average yields a strongly anisotropic ($v_1 \gg v_2$) *algebraic* decay,

$$F_A(\mathbf{r}) \propto |(v_1 r_1)^2 + (v_2 r_2)^2|^{-\frac{2g_A}{\pi}}, \quad (12)$$

implying that eigenstates of the non-Hermitian operator \hat{H} are *critical* [7,8]. Moreover, since the algebraic decay of F_A is not influenced by $\text{Im } z$, we infer that the critical nature of the wavefunctions is also insensitive to $\text{Im } z$.

As for the non-abelian sector, the non-trivial relationship between the fields B_μ and the effective ‘gauge’ field, $\tilde{\xi}$ makes the calculation of the correlation function, F_B more involved (c.f. Refs. [5–7,10]). However, although at present no rigorous statements on the long distance behavior of the correlation function F_B can be made in general, insight can be drawn from the following facts: a) The theory possesses a strong coupling fixed point described by the Wess-Zumino-Novikov-Witten theory $SU_4(2)$ [5,7] (corresponding to an infinite strength of the non-abelian components of the disorder potential), and b) in this limit [7,10], the correlation function,

$$F_B(\mathbf{r}) \propto |(v_1 r_1)^2 + (v_2 r_2)^2|^{-1/2} \quad (13)$$

is again algebraic. Note that the scaling exponent in Eq. (13) is fixed solely by the number of nodes (two) in Eq. (4). Therefore, given that the wavefunctions are critical in the strongly disordered limit, it seems highly plausible that they remain critical in general.

In this Letter, we have studied the spectral properties of the two-dimensional random Schrödinger operator in the presence of a uniform imaginary vector potential, \mathbf{h} . Mapping it to a Hermitian Hamiltonian with chiral symmetry, a gradient expansion identifies properties of the non-Hermitian operator with those of a stochastic Dirac Hamiltonian $\hat{\mathcal{H}}_D$. This correspondence allows for the explicit construction of eigenfunctions of the non-Hermitian Hamiltonian for individual realizations of the disorder. In the thermodynamic limit, the wavefunctions were shown to be delocalized along *both* the directions parallel *and* perpendicular to \mathbf{h} .

Finally we note that the main characteristics of the wavefunction (11), long-ranged disorder dependence encoded in the ‘gauge fields’, and exponential amplification of the wavefunction modulus, can manifest themselves in more complex phenomena than that discussed in this Letter: Firstly, one can envisage systems with non-trivial topology and/or stochastic potentials with superimposed regular structures (e.g. spatially non-uniform distribution functions). In such cases the behavior of the wavefunctions may change *qualitatively* (e.g. to localization). Secondly, we note that the sensitivity of the modulus to disorder results in *strong statistical fluctuations* of the wavefunctions which are large in comparison to the average. In particular, correlation functions of the moments, C acquire scaling exponents with non-linear dependence in q_i , a characteristic related to multi-fractality.

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